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ON THE TAYLOR APPROXIMATION OF CONTROL SYSTEMS(U)
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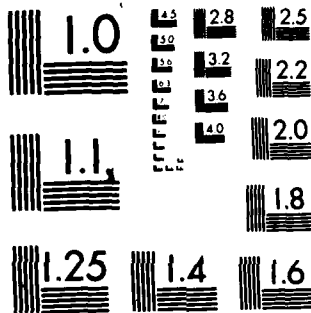
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ON THE TAYLOR APPROXIMATION OF
CONTROL SYSTEMS

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ABSTRACT

Let g_i ($i = 1, \dots, m$) be smooth vector fields on \mathbb{R}^d , and let $T^{n-1} g_i$ be their Taylor expansions of order $n - 1$ at the origin. The system

$$\dot{x}(t) = \sum_{i=1}^m g_i(x(t)) u_i(t), \quad x(0) = 0 \in \mathbb{R}^d$$

generates an input-output map

$u(\cdot) \mapsto x(u, \cdot)$ whose n -th order Taylor approximation $x_n(u, \cdot)$ can be

obtained by computing the n -th Picard iterate for the reduced system

$$\dot{x}(t) = \sum_{i=1}^m (T^n g_i)(x(t)) u_i(t), \quad x(0) = 0,$$

discarding the terms of order

$> n$. For $z \in \mathbb{R}^d$, directional error bounds of the form

$$| \langle z, x(u, t) - x_n(u, t) \rangle | \leq C t^{n-p} \left| \int_0^t \left| \int_0^s \sum_{i=1}^m g_i(0) u_i(\sigma) d\sigma \right|^p ds \right|$$

can be given. These estimates improve those supplied by the classical Taylor's theorem and yield results concerning local non-controllability.

AMS (MOS) Subject Classifications: 93C10, 49E15

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ON THE TAYLOR APPROXIMATION OF CONTROL SYSTEMS

Alberto Bressan

Consider an autonomous nonlinear system on \mathbb{R}^d with control entering linearly:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + \sum_{i=1}^m g_i(x(t)) u_i(t) \\ x(0) &= 0, \quad t > 0. \end{aligned} \tag{1.1}$$

Systems of the form (1.1) were first studied in a paper by Hermes and Haynes [4] and received continued attention since then. If f and the g_i 's are smooth vector fields, then (1.1) yields a smooth input-output map $\phi: L^1([0, \infty); \mathbb{R}^m) \rightarrow C([0, \infty); \mathbb{R}^d)$ defined by $\phi(u)(t) = x(u, t)$, where $x(u, \cdot)$ is the unique solution of (1.1) corresponding to the control u . In general there exist no explicit formulas giving the trajectory $x(u, \cdot)$ directly in terms of the control. It is therefore natural to look for some computable approximation of the map ϕ . The Taylor expansion of ϕ in terms of Volterra kernels was considered by Brockett [1]. The local approximation of a control system of the form

$$\dot{x}(t) = \sum_{i=1}^m g_i(x(t)) u_i(t), \quad x(0) = 0 \tag{1.2}$$

by means of an auxiliary (nilpotent) system is studied in [5]. For analytic systems, expansions in formal power series are given in [3].

In the present paper we approximate the input-output map ϕ generated by (1.2) with linear combinations of certain iterated integrals, here called integral monomials. Using functional analytic techniques, we derive a simple procedure to recursively compute the coefficients of the Taylor expansion for ϕ in terms of the Taylor coefficients of the g_i 's at the origin. No analyticity assumptions are needed. Our main concern is a

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precise estimate of the approximation errors. Since the system (1.2) is linear in u , these errors are essentially due to the nonlinearity of the vector fields g_i , which depend only on x . We thus obtain bounds which depend not on the size of the control $u(\cdot)$ but only on the norm of a first order approximation to the trajectory $x(u, \cdot)$. The estimates contained in Theorems 1 and 2, in sections 5 and 6 respectively, improve those supplied by the classical Taylor's theorem and are sharp enough to yield results on local non-controllability for the system (1.1). Examples are provided in sections 4 to 6. Some related results recently appeared in [6].

§ 2. Preliminary Abstract Results

Given two Banach spaces E and F , $k \geq 0$, we denote by $L^k(E, F)$ the space of continuous k -linear mappings Λ from $\otimes_k E = E \otimes E \otimes \dots \otimes E$ (k times) into F , with the operator norm

$$\|\Lambda\| = \sup \{ \|\Lambda(v_1, \dots, v_k)\|_F; \|v_i\|_E < 1, i = 1, \dots, k \}.$$

If $\phi: E \rightarrow F$ is a smooth mapping, its k -th Frechet derivative at a point $u \in E$ is $D^k \phi(u) \in L^k(E, F)$. It is well known that high-order derivatives are symmetric multilinear mappings. $D^k \phi(u)$ is therefore completely determined by assigning its values on elements of the form $v^{[k]} = (v, v, \dots, v) \in \otimes_k E$. We write B_ρ for the closed ball centered at the origin with radius ρ , and $T^n \phi$ for the n -th order Taylor expansion of the map ϕ at the origin, i.e.

$$(T^n \phi)(u) = \sum_{i=0}^n \frac{1}{i!} D^i \phi(o) \cdot u^{[i]}.$$

Given a function $\phi = \phi(u, x)$ defined on a product space $E \times F$, its partial derivatives are denoted by ∂_u, ∂_x . In this case, $T^n \phi$ stands for the n -th order Taylor expansion of ϕ at $(0, 0)$. The Landau order symbols O and o will also be used. For the basic properties of differential calculus in Banach spaces, our general reference is Dieudonne [2]. The approximation procedure considered in this paper is based on the following simple result:

Proposition 1. Let E, F be Banach spaces, let $(u, x) \rightarrow \Phi(u, x)$ be a C^k map from a neighborhood of the origin in $E \times F$ into F such that $\Phi(0, 0) = 0$, $\partial_x \Phi(0, 0) = 0$, and let the map $u : E \rightarrow F$ be implicitly defined by $\phi(u) = \Phi(u, \phi(u))$. If Ψ, ψ are C^k maps such that $T^n \Psi = T^n \Phi$ and $T^{n-1} \psi = T^{n-1} \phi$ for some n , $0 < n < k$, then the n -th order Taylor expansion at the origin of the map $u \rightarrow \Psi(u, \psi(u))$ coincides with $T^n \phi$.
Proof. Since the maps $\phi(\cdot)$ and $\Psi(\cdot, \psi(\cdot))$ are both C^k , we only need to show that, under the above hypothesis,

$$\lim_{u \rightarrow 0} \|\phi(u) - \Psi(u, \psi(u))\| \cdot \|u\|^{-n} = 0. \quad (2.1)$$

By Taylor's theorem, for a suitable constant C and for u, x sufficiently close to the origin one has

$$\begin{aligned} \|\phi(u)\| &\leq C\|u\|, \|\psi(u)\| \leq C\|u\|, \|\phi(u) - \psi(u)\| \leq C\|u\|^n, \\ \|\partial_x \Psi(u, x)\| &\leq C(\|u\| + \|x\|), \|\Phi(u, x) - \Psi(u, x)\| \leq C(\|u\|^{n+1} + \|x\|^{n+1}). \end{aligned}$$

These inequalities imply

$$\|\phi(u) - \Psi(u, \psi(u))\| \leq \|\Phi(u, \phi(u)) - \Psi(u, \phi(u))\| + \|\Psi(u, \phi(u)) - \Psi(u, \psi(u))\|$$

$$\leq C(\|u\|^{n+1} + \|\phi(u)\|^{n+1}) + \int_0^1 \|\partial_x \Psi(u, \xi\phi(u) + (1-\xi)\psi(u))\| \cdot \|\phi(u) - \psi(u)\| d\xi$$

$$\leq C(1+C^{n+1})\|u\|^{n+1} + C(\|u\| + C\|u\|) \cdot C\|u\|^n,$$

which proves (2.1).

Corollary 1. If Φ, ϕ satisfy the assumptions made in Proposition 1, then

$$\lim_{u \rightarrow 0} \|\phi(u, (T^n \phi)(u)) - (T^n \phi)(u)\| \cdot \|u\|^{-n} = 0. \quad (2.2)$$

§3 The Taylor Formula

Most of this paper is concerned with the system (1.2). Notice that, by setting $u_1 \equiv 1$, a control system of the form (1.1) is recovered as a special case of (1.2). To simplify all further discussion, we assume that the vector fields g_i are C^∞ . In the following, $|g_1(x)|$ denotes the euclidean norm of the vector

$g_i(x) \in \mathbb{R}^d$, while $|\nabla g_i(x)|$ is the operator norm of the $d \times d$ matrix of first order partial derivatives of g_i at x . The set of admissible controls is defined as

$$U = \{u = (u_1, \dots, u_m) \in L^1([0, \infty); \mathbb{R}^m); |u_i(t)| < 1, i = 1, \dots, m, t > 0\}.$$

The L^1 norm on the set of controls and the C^0 norm on the set of trajectories will be always used. We write (1.2) in the more compact form

$$\dot{x}(t) = G(x(t))u(t), \quad x(0) = 0. \quad (3.1)$$

G is therefore a C^∞ map from \mathbb{R}^d into $L(\mathbb{R}^m, \mathbb{R}^d)$. Call $G^{(j)}(z)$ its j -th derivative at $z \in \mathbb{R}^d$ and let $G_n = T^n G$. Set $E = L^1([0, \infty); \mathbb{R}^m)$, $F = C([0, \infty); \mathbb{R}^d)$, and define the map $\Phi : E \times F \rightarrow F$ by

$$\Phi(u, y)(t) = \int_0^t G(y(s))u(s) ds. \quad (3.2)$$

Then $\Phi = \Phi'' \circ \Phi'$, with $\Phi'(u, y)(t) = (u(t), G(y(t)))$

and

$$\Phi''(u, z)(t) = \int_0^t Z(s)u(s)ds.$$

Φ' is a C^∞ substitution operator and Φ'' is bilinear. Hence Φ is k times continuously Fréchet differentiable, for all k . In particular, $\partial_x^j \Phi(u_0, x_0)(t)$ is the j -linear map

$$y^{[j]} \mapsto \int_0^t G^{(j)}(x_0(s)) y^{[j]}(s) u_0(s) ds, \quad (3.3)$$

$\partial_u \partial_x^j \Phi(u_0, x_0)(t)$ is the multilinear map

$$(u, y^{[j]}) \mapsto \int_0^t G^{(j)}(x_0(s)) y^{[j]}(s) u(s) ds \quad (3.4)$$

and $\partial_u^i \partial_x^j \Phi \equiv 0$ for $i > 1$, because the dependence on u is linear. Notice that the input-output map $u(\cdot) \mapsto x(u, \cdot) = \Phi(u)(\cdot)$ generated by (1.2), or equivalently by (3.1), is implicitly defined by the equation $\Phi(u) = \Phi(u, \Phi(u))$, and that both Φ and $\partial_x \Phi$ vanish at $(0, 0)$. The Taylor expansion of Φ at the origin can therefore be computed recursively, by means of Proposition 1. We will show that $T^n \Phi$ can always be written as a finite sum of certain iterated integrals of the control u , here called integral monomials (integromials, in short).

Definition. The family $M = M(m, d)$ of integral monomials is the smallest set of mappings μ from $L^1([0, \infty); \mathbb{R}^m)$ into \mathbb{R}^d with the following properties:

i) For every linear $A : \mathbb{R}^m \rightarrow \mathbb{R}^d$, the map

$$\mu : (u, t) \rightarrow \int_0^t A \cdot u(s) ds \quad (3.5)$$

is in M

ii) If $\mu_1, \dots, \mu_k \in M$ and if $B : (\mathbb{R}^d)^k \rightarrow L(\mathbb{R}^m; \mathbb{R}^d)$ is k -linear, then M also contains the map

$$\mu : (u, t) \rightarrow \int_0^t B(\mu_1(u, s), \dots, \mu_k(u, s)) u(s) ds. \quad (3.6)$$

Using a canonical identification, we shall regard integral monomials as multilinear mappings from $L^1([0, \infty); \mathbb{R}^m)$ into $C^0([0, \infty); \mathbb{R}^d)$. If $\mu \in M$ is k -linear, we say that μ has order k . Notice that if in (3.6) μ_i has order v_i ($i = 1, \dots, k$), then μ has order $1 + v_1 + \dots + v_k$. Consider now the approximate system

$$\dot{x}(t) = G_v(x(t))u(t), \quad x(0) = 0, \quad (3.7)$$

recalling our definition : $G_v = T^v G$. The first Picard iterate for (3.2)

$$P_1(u, t) = \int_0^t G(0)u(s) ds \quad (3.8)$$

is an integral monomial of the form (3.5). Moreover, if the n -th Picard iterate P_n can be written as a finite sum of integral monomials, say

$$P_n(u, t) = \sum_{j=1}^{N(n)} \mu_j(u, t),$$

then the same holds for P_{n+1} . Indeed

$$P_{n+1}(u, t) = \int_0^t \sum_{k=0}^v \frac{1}{k!} G^{(k)}(0) (P_n(u, s))^{[k]} u(s) ds$$

can be reexpanded into a finite sum of integral monomials of the form (3.6), namely

$$P_{n+1}(u, t) = \sum_{k=0}^v \sum_{\sigma \in \Gamma} \int_0^t \frac{1}{k!} G^{(k)}(0) (\mu_{\sigma(1)}(u, s), \dots, \mu_{\sigma(k)}(u, s)) u(s) ds, \quad (3.9)$$

where $\Gamma = \Gamma(k, N(n))$ is the set of all mappings σ from $\{1, 2, \dots, k\}$ into $\{1, \dots, N(n)\}$.

In practice, the k -th order Taylor expansion $T^k \phi$ at the origin for the input-output map $\phi : u(\cdot) \rightarrow x(u, \cdot)$ generated by (3.1) can be obtained by either of the following methods:

I) Compute the k-th Picard iterate for the system

$$\dot{x}(t) = G_{k-1}(x(t))u(t), \quad x(0) = 0 \quad (3.10)$$

and discard the integral monomials of order $> k$.

II) Set $x_n(u, \cdot) = (T^n \phi)(u)(\cdot)$ and recursively derive x_n from x_{n-1} ($n = 1, \dots, k$) by expanding the map

$$(u, t) \rightarrow \int_0^t G_{n-1}(x_{n-1}(u, s)) u(s) ds \quad (3.11)$$

into a sum of integral monomials, discarding those which have order $> n$.

Indeed, for $n > 1$ set

$$\phi_n(u, x)(t) = \int_0^t G_{n-1}(x(s)) u(s) ds \quad (3.12)$$

and check that the Taylor expansions at $(0, 0)$ of ϕ_n and ϕ (defined at (3.2)) coincide

up to order n . Let P_n be the n -th Picard iterate for (3.10), regarded as a map

$u(\cdot) \rightarrow P_n(u, \cdot)$ from L^1 into C^0 . Assume that $T^{n-1}P_{n-1} = T^{n-1}\phi$. Then, by setting

$\Psi = \phi_k, \psi = P_{n-1}$, Proposition 1 yields $T^n P_n = T^n \phi$, provided $n \leq k$. By induction,

$T^k P_k = T^k \phi$. Observing that P_k is a finite sum of integromials, to obtain its k -th

order Taylor expansion at the origin one merely discards the terms of order $> k$. This

justifies I). Now set $\Psi = \phi_n, \psi = T^{n-1}\phi$. By Proposition 1 the map

$u \rightarrow \phi_n(u, (T^{n-1}\phi)(u))$, otherwise defined at (3.11), has the same n -th order Taylor

expansion at the origin as ϕ . Discarding from (3.11) the integral monomials of

order $> n$ we thus recover $T^n \phi$ from $T^{n-1}\phi$. One obtains $T^k \phi$ by repeating the above procedure for $n = 1, \dots, k$.

§4 Examples

Example 1. The third order Taylor expansion for the scalar system

$$\dot{x}(t) = \cos x(t)u_1(t) + u_2(t), \quad x(0) = 0$$

is

$$\begin{aligned} (T^3 \phi)(u)(t) = & \int_0^t [u_1(s) + u_2(s)] ds - \frac{1}{2} \int_0^t u_1(\sigma_1) \left(\int_0^{\sigma_1} u_1(\sigma_2) d\sigma_2 \right)^2 d\sigma_1 \\ & - \int_0^t u_1(\sigma_1) \left(\int_0^{\sigma_1} u_1(\sigma_2) d\sigma_2 \right) \left(\int_0^{\sigma_1} u_2(\sigma_2) d\sigma_2 \right) d\sigma_1 - \frac{1}{2} \int_0^t u_1(\sigma_1) \left(\int_0^{\sigma_1} u_2(\sigma_2) d\sigma_2 \right)^2 d\sigma_1. \end{aligned}$$

If u_1 is constrained to be identically 1 we obtain an approximation for the system

$$\dot{x}(t) = \cos x(t) + u(t), \quad x(0) = 0, \quad \text{namely}$$

$$x(u, t) = t - \frac{t^3}{6} + \frac{1}{2} \int_0^t u(\sigma) d\sigma - \frac{1}{2} \int_0^t \left(\int_0^{\sigma} u(\sigma_2) d\sigma_2 \right)^2 d\sigma_1 \\ - \int_0^t \sigma_1 \int_0^{\sigma} u(\sigma_2) d\sigma_2 d\sigma_1 + 0(t^4).$$

Example 2. Consider on \mathbb{R}^3 the control system

$$\dot{x} = X(x) + Y(x) \cdot u, \quad x(0) = 0. \quad (4.1)$$

Assume that $X(0) = 0$ and $\text{span} \{(\text{ad}^v X, Y); v = 0, 1, 2\} = \mathbb{R}^3$. Then the map $w : (s_1, s_2, s_3) \rightarrow (\exp s_1 Y) \cdot (\exp s_2 [Y, X]) \cdot (\exp s_3 [[Y, X], X]) (0)$ defines a local chart of a neighborhood of the origin in \mathbb{R}^3 . In this chart, the third order Taylor approximation for (4.1) takes the form

$$x_1(u, t) = \int_0^t u(s) ds + \frac{k_1}{2} \int_0^t \left(\int_0^{\sigma} u(\sigma_2) d\sigma_2 \right)^2 d\sigma_1 + 0(t^4) \\ x_2(u, t) = \int_0^t \int_0^{\sigma} u(\sigma_2) d\sigma_2 d\sigma_1 + \frac{k_2}{2} \int_0^t \left(\int_0^{\sigma} u(\sigma_2) d\sigma_2 \right)^2 d\sigma_1 + 0(t^4) \\ x_3(u, t) = \int_0^t \int_0^{\sigma} \int_0^{\sigma_2} u(\sigma_3) d\sigma_3 d\sigma_2 d\sigma_1 + \frac{k_3}{2} \int_0^t \left(\int_0^{\sigma} u(\sigma_2) d\sigma_2 \right)^2 d\sigma_1 + 0(t^4), \quad (4.2)$$

where the constants k_1, k_2, k_3 are obtained from the linear relation

$$k_1 Y(0) + k_2 [Y, X](0) + k_3 [[Y, X], X](0) = [Y, [Y, X]](0).$$

To prove (4.2), we need to compute some Taylor coefficients of X and Y in the given chart. The definition of w implies that $Y \equiv (1, 0, 0)$, $[Y, X](0) = (0, 1, 0)$ and $[[Y, X], X](0) = (0, 0, 1)$. We thus have

$$\frac{\partial X}{\partial s_1}(0) = Y \cdot X(0) = [Y, X](0) = (0, 1, 0)$$

$$\frac{\partial X}{\partial s_2}(0) = [Y, X] \cdot X(0) = [[Y, X], X](0) = (0, 0, 1)$$

$$\frac{\partial^2 X}{\partial s_1^2}(0) = Y(Y \cdot X)(0) = [Y, [Y, X]](0) = (k_1, k_2, k_3).$$

One can now multiply the vector field X by a fictitious control $u_0 \equiv 1$ and use Theorem 1 to get (4.2).

Remark: It is clear that the truncated Taylor expansions are not invariant under change of coordinates. The above example shows how the local Lie algebraic structure of the system can yield a chart in which the expansion takes the simplest possible form. For a general procedure to construct "canonical" charts, see [5 p. 127].

§5 Error bounds.

High-order Taylor expansions are the primary tool in the local study of functionals at points of singularity. If the origin is a stationary point for a map $\Lambda : E \rightarrow \mathbb{R}$, one classically proves that Λ attains a local minimum at 0 by showing that for some $n > 1$

$$(T^n \Lambda)(u) - \Lambda(0) > C \cdot \|u\|^n \quad (5.1)$$

for all u in a neighborhood of the origin. One would hope to use a similar argument and prove non-controllability results concerning the system (1.1). For example, let $f(0) = 0$. Then (1.1) is not locally controllable at the origin if, for some nontrivial $z^* \in \mathbb{R}^d$, $T > 0$, the functional $\Lambda(u) = \langle z^*, x(u, T) \rangle$ attains a local minimum at $\bar{u} \equiv 0$.

Coercivity conditions such as (5.1), however, can seldom be obtained in connection with control systems. To overcome this difficulty, one needs bounds on the error

$$\varepsilon(u) = \|(T^n \phi)(u) - \phi(u)\| \text{ which are sharper than the classical bound } \varepsilon(u) < C\|u\|^{n+1}$$

supplied by Taylor's theorem.

Lemma 1. Let $x(u, \cdot)$ be the solution of (1.2) corresponding to the control u . Then there exists a $t_0 > 0$ such that, for every continuous map γ from $[0, t_0]$ into the unit ball $B_1 \subseteq \mathbb{R}^d$,

$$|y(t) - x(u, t)| \leq 2 \sup \left\{ \left| \int_0^t G(y(s))u(s)ds - y(\tau) \right|, 0 \leq \tau \leq t \right\} \quad (5.2)$$

for all $t \in [0, t_0]$, $u \in U$.

Proof. Set $t_0 = M^{-1}$, with

$$M = 2m \cdot \sup \{ |g_i(x)| + |\nabla g_i(x)|, x \in B_1, i = 1, \dots, m \}.$$

Then the map Φ defined at (3.2) satisfies

$$\|\Phi(u, y_1) - \Phi(u, y_2)\|_{C[0, t_0]} \leq \frac{1}{2} \|y_1 - y_2\|_{C[0, t_0]}$$

and $\Phi(u, y_1)(t) \in B_1$, for every $u \in U$, $t \in [0, t_0]$ and every continuous maps

$y_1, y_2 : [0, t_0] \rightarrow B_1$. Since $x(u, \cdot)$ is a fixed point for the map $y \rightarrow \Phi(u, y)$, the contraction mapping theorem yields $\|y - x(u, \cdot)\| \leq 2\|y - \Phi(u, y)\|$, i.e., (5.2).

Lemma 2. For each $u \in U$, let $x(u, \cdot)$ be the solution of (3.1) and let $y_n(u, \cdot)$ be the solution of

$$\dot{y}(t) = G_{n-1}(y(t))u(t), \quad x(0) = 0. \quad (5.3)$$

Then there exist C and $t_0 > 0$ such that

$$|y_n(u, t) - x(u, t)| \leq C \int_0^t \left| \int_0^s G(u)u(\sigma)ds \right|^n ds \quad (5.4)$$

for all $t \in [0, t_0]$, $u \in U$.

Proof. Choose $t_1 > 0$ so small that $y_n(u, t) \in B_1$ for all $u \in U$, $t \in [0, t_1]$. Lemma 1 implies the existence of a $t_0 > 0$ such that $t_0 \leq t_1$ and

$$|y_n(u, t) - x(u, t)| \leq 2 \sup \left\{ \left| \int_0^t G(y_n(u, s))u(s)ds - y_n(u, \tau) \right|, 0 \leq \tau \leq t \right\} \quad (5.5)$$

for all $t \in [0, t_0]$, $u \in U$. Using (5.3) we have

$$\begin{aligned} |y_n(u, t) - x(u, t)| &\leq 2 \sup \left\{ \left| \int_0^t [G(y_n(u, s)) - G_{n-1}(y_n(u, s))]u(s)ds \right|, 0 \leq \tau \leq t \right\} \\ &\leq 2 \cdot C_1 \int_0^t |y_n(u, s)|^n ds, \end{aligned} \quad (5.6)$$

where the constant C_1 depends only on the size of the n -th derivative of G on B_1 .

Set

$$M = m \cdot \sup \{ |\nabla(T^{n-1}g_1)(z)|, z \in B_1 \subset \mathbb{R}^d, i = 1, \dots, m \},$$

$$n(t) = \left| \int_0^t G(u)u(s)ds \right|, \quad w(t) = |y_n(u, t) - \int_0^t G(u)u(s)ds|. \quad \text{For almost every}$$

$$t \in [0, t_0], \quad \dot{w}(t) \leq |G_{n-1}(y_n(u, t)) - G_{n-1}(0)| \leq M \cdot |y_n(u, t)| \leq M(w(t) + n(t)).$$

Since $w(0) = 0$, Gronwall's lemma yields $w(t) \leq C_2 \int_0^t n(s) ds$, with $C_2 = Me^{Mt_0}$. Using Holder's inequality we get the estimate

$$\begin{aligned} 2 \int_0^t |y(u,s)|^n ds &\leq 2 \int_0^t [n(\tau) + C_2 \int_0^\tau n(s) ds]^n d\tau \\ &\leq \int_0^t (2n(\tau))^n d\tau + \int_0^t (2C_2 \int_0^\tau n(s) ds)^n d\tau \\ &\leq \int_0^t (2n(\tau))^n d\tau + (2C_2)^n t \left(\int_0^t n^n(s) ds \right) t^{n-1} \leq C_3 \int_0^t n^n(\tau) d\tau, \end{aligned} \quad (5.7)$$

with $C_3 = 2^n + (2C_2 t_0)^n$. Combining (5.6) with (5.7) one has (5.3), with $C = C_1 C_3$.

The above lemmas provide a simple estimate for the error in the Taylor expansion of ϕ .

Theorem 1. Let $x(u, \cdot)$ be the solution of (3.1). Set $G_{n-1} = T^{n-1}G$,

$x_n(u, t) = (T^n \phi)(u)(t)$. Then there exist $C, t_0 > 0$ such that

$$\begin{aligned} |x(u, t) - x_n(u, t)| &\leq C \int_0^t \left| \int_0^s G(o)u(\sigma) d\sigma \right|^n ds + \\ &+ 2 \sup \left\{ |x_n(u, t) - \int_0^t G_{n-1}(x_n(u, s))u(s) ds|; 0 \leq \tau \leq t \right\}, \end{aligned} \quad (5.8)$$

for every $u \in U, t \in [0, t_0]$.

Indeed, $|x(u, t) - x_n(u, t)| \leq |x(u, t) - y_n(u, t)| + |y_n(u, t) - x_n(u, t)|$. The estimates (5.4) and (5.2) with G replaced by G_{n-1} yield (5.8).

Example 3. Consider the bidimensional system $\frac{d}{dt}(x, y) = (u+y \cos x, x^2 - x^3)$,

$(x(0), y(0)) = (0, 0)$. Its third order approximation is

$$x(u, t) = \int_0^t u(s) ds + \varepsilon_1(u, t), \quad y(u, t) = \int_0^t \left(\int_0^s u(\sigma) d\sigma \right)^2 ds + \varepsilon_2(u, t).$$

Theorem 2 yields the bounds

$$|\varepsilon_1(u, t)| \leq C \int_0^t \left| \int_0^s u(\sigma) d\sigma \right|^3 ds + 2 \int_0^t \int_0^1 \left(\int_0^2 u(\sigma_3) d\sigma_3 \right)^2 d\sigma_2 d\sigma_1, \quad (5.9)$$

for $i = 1, 2$, $u \in U$ and t sufficiently small. From (5.9) we see that

$|\varepsilon_2(u, t)| = o(\int_0^t (\int_0^s u(\sigma) d\sigma)^2 ds)$, hence, for small t , $x(u, t) > 0$ and the system is not locally controllable. For any fixed $t_0 > 0$, consider the control $u_\lambda(t) = \cos \lambda t$. As $\lambda \rightarrow \infty$, $\|u_\lambda\|_{L^1[0, t_0]}^4$ tends to $(2 t_0 / \pi)^4$, while $\int_0^t (\int_0^s u_\lambda(\sigma) d\sigma)^2 ds$ tends to zero. Bounds of the type $|\varepsilon_1(u, t)| < Ct^4$ or $|\varepsilon_1(u, t)| < C \cdot \|u\|^4$ are therefore too weak for proving non-controllability even in this simple case.

§6 Directional Estimates.

To obtain more precise bounds on the error in the Taylor expansion of (1.2), in this section we split \mathbb{R}^d into a sum of orthogonal subspaces V_p and estimate the size of the error separately on each V_p . Given the control system (1.2), define an increasing sequence of subspaces $S_p \subseteq \mathbb{R}^d$ recursively by setting i) $S_0 = \{0\}$, ii) S_p is the smallest subspace of \mathbb{R}^d such that for all $i = 1, \dots, m$ and $k = 0, \dots, p$ one has $D^k g_i(0)(z_1, \dots, z_k) \in S_p$ for every k -tuple (z_1, \dots, z_k) with $z_i \in S_{j_i}$ and $j_1 + j_2 + \dots + j_k \leq p$. In particular, S_1 is the smallest subspace that contains the m vectors $g_i(0)$ and is invariant under the linear operators $Vg_i(0)$ ($i = 1, \dots, m$). Now choose any $\bar{p} > 1$. For $1 \leq p < \bar{p}$ define V_p as the orthogonal complement of S_{p-1} in S_p . Let $V_{\bar{p}}$ be the orthogonal complement of $S_{\bar{p}-1}$ in \mathbb{R}^d . Finally, denote by π_p the orthogonal projection $\mathbb{R}^d \rightarrow V_p$. With this notation, we have

Theorem 2. Let $u(\cdot) \rightarrow x(u, \cdot)$ be the input-output map generated by (1.2) and let $u(\cdot) \rightarrow x_n(u, \cdot)$ be its n -th order Taylor approximation about the null control. If $p \in \{1, \dots, \bar{p}\}$ and $n > p$, then there exist $C_0, t_0 > 0$ such that

$$|\pi_p(x(u, t) - x_n(u, t))| < C_0 t^{n-p} \int_0^t \left| \int_0^s G(0) u(\sigma) d\sigma \right|^p ds \quad (6.1)$$

for all $t \in [0, t_0]$, $u \in U$.

Proof. Let y_n be the solution of (5.3). By Lemma 2, the difference $y_n(u, t) - x(u, t)$ satisfies a bound of the form (6.1) for any $p = 1, \dots, \bar{p}$. Therefore we only need to show that

$$|\pi_p(y_n(u,t) - x_n(u,t))| \leq C_0 t^{n-p} \int_0^t |x_1(u,s)|^p ds,$$

where

$$x_1(u,s) = \int_0^s G(o)u(\sigma) d\sigma$$

gives the first order Taylor approximation to the trajectory $x(\cdot, \cdot)$. In §3, x_n was proven to be a finite sum of integral monomials occurring within the first n Picard iterates for the system (5.3). To keep track of its size, to each one of the above integromials we attach an integer $\gamma(\mu)$ as follows: If $\mu = x_1$, defined at (6.3), set $\gamma(\mu) = 1$. If

$$\mu(u,t) = \int_0^t \frac{1}{k!} G^{(k)}(o)(\mu_1(u,s), \dots, \mu_k(u,s))u(s) ds, \quad (6.4)$$

set $\gamma(\mu) = \gamma(\mu_1) + \dots + \gamma(\mu_k)$. We stress that this definition refers exclusively to the integromials arising via Picard iterations for the particular system (5.3) presently considered. Comparing the definitions of γ and of the subspaces S_p , it is clear that $\mu(u,t) \in S_{\gamma(\mu)}$ for all $t > 0, u \in U$. Also notice that if the order of μ is $v > 1$, then $\gamma(\mu) \leq v - 1$. A basic estimate on the size of integral monomials is now given.

Lemma 3. Let μ be an integral monomial of order $v > 1$ occurring in some Picard iterate for (5.3), and let $\gamma(\mu) = p$. Then there exists a constant C such that

$$|\mu(u,t)| \leq C t^{v-p-1} \int_0^t |x_1(u,s)|^p ds \quad (6.5)$$

for all $u \in U, 0 \leq t \leq 1$.

Proof. Notice first that for the integromial $x_1(\cdot, \cdot)$ we have $v = p = 1$. For all others, $v > 1$. Moreover, the only integral monomials for which $v = p + 1$ have the form

$$\mu(u,t) = \int_0^t \frac{1}{p!} G^{(p)}(o)(x_1(u,s))^{[p]} u(s) ds \quad (6.6)$$

and clearly satisfy an estimate of the type (6.5). The general case will be proved by induction on v , assuming $v \geq p + 2$. If μ is given by (6.4), let μ_i have order $v_i < v$ and let $\gamma(\mu_i) = p_i$ ($i = 1, \dots, k$). The inductive hypothesis implies that either $\mu_i = x_1$ or

$$|\mu_i(u,s)| \leq C_i s^{v_i-p_i-1} \int_0^s |x_1(u,\sigma)|^{p_i} d\sigma \quad (6.7)$$

for some constant C_1 and all $u \in U$, $0 < t \leq 1$.

Using the symmetry of $G^{(k)}(0)$, we can reorder the μ_i and assume that (6.4) has the form

$$\mu(u, t) = \int_0^t \frac{1}{k!} G^{(k)}(0)(\mu_1(u, s), \dots, \mu_h(u, s), x_1(u, s), \dots, x_1(u, s)) u(s) ds \quad (6.8)$$

for some $0 < h \leq k$, with each μ_i ($1 \leq i \leq h$) having order $v_i > 1$. The order of μ in (6.8) is then $v = 1 + v_1 + \dots + v_h + k - h$, while $p = \gamma(\mu) = p_1 + \dots + p_h + k - h$. In the following, we set $q = p_1 + \dots + p_h = p - k + h$. If $q = 0$, then μ has the form

(6.6) and the estimate (6.5) is immediate. If $q > 0$, the inductive hypothesis and

Holder's inequality imply

$$\begin{aligned} |\mu(u, t)| &\leq C' \int_0^t |x_1(u, s)|^{k-h} \prod_{i=1}^h [C_1 s^{v_i - p_i - 1} \int_0^s |x_1(u, \sigma)|^{p_i} d\sigma] ds \\ &\leq C \int_0^t s^{(v-1)-p-h} \prod_{i=1}^h \left[\left(\int_0^s |x_1(u, \sigma)|^q d\sigma \right)^{p_i/q} \cdot s^{(q-p_i)/q} \right] ds \\ &\leq C \int_0^t s^{v-p-2} |x_1(u, s)|^{k-h} \left(\int_0^s |x_1(u, \sigma)|^q d\sigma \right) ds, \end{aligned} \quad (6.9)$$

where the constants C, C', C_1 are independent of u and t . If $h = k$, (6.5) is a trivial consequence of (6.9). If $h < k$, one recovers again (6.5) from (6.9) integrating by parts and using Holder's inequality:

$$\begin{aligned} |\mu(u, t)| &\leq C t^{v-p-2} \left(\int_0^t |x_1(u, s)|^{k-h} ds \right) \left(\int_0^t |x_1(u, s)|^q ds \right) \\ &\leq C t^{v-p-1} \int_0^t |x_1(u, s)|^{k-h+q} ds. \end{aligned}$$

Returning to the proof of Theorem 2, for any control $u \in U$ and any constants

$$C, \tau > 0, \text{ define } \lambda_u(C, \tau) \text{ as the set of all maps } z \in C([0, \tau]; \mathbb{R}^d) \text{ such that} \quad (6.10)$$

$$|z_p(z(t)) - x_n(u, t)| \leq C \cdot t^{n-p} \int_0^t |x_1(u, s)|^p ds$$

for all $t \in [0, \tau]$, $p = 1, \dots, \bar{p}$. We claim that $y_n(u, \cdot) \in \lambda_u(C_0, t_0)$ for some $C_0, t_0 > 0$ and all $u \in U$.

Lemma 4. There exist constants C_0, t_0 such that, for every $u \in U$, the map ψ_u defined by

$$\psi_u(z)(t) = \int_0^t G_{n-1}(z(s)) u(s) ds$$

maps $\lambda_u(C_0, t_0)$ into itself.

Proof. Given $u \in U$, $z \in C([0, \tau]; \mathbb{R}^d)$, for $p = 1, \dots, \bar{p}$ set

$w_p(t) = \pi_p(z(t) - x_n(u, t))$. We then have

$$\begin{aligned} |\pi_p(x_n)(u, t) - \psi_u(z(t))| &\leq |\pi_p(x_n(u, t) - \int_0^t G_{n-1}(x_n(u, s))u(s)ds)| \\ &+ |\pi_p(\int_0^t G_{n-1}(x_n(u, s))u(s)ds - \int_0^t G_{n-1}(x_n(u, s) + \sum_{p=1}^{\bar{p}} w_p(s))u(s)ds)| \\ &= |A(u, t)| + |B(u, t)|. \end{aligned} \quad (6.11)$$

Notice that $A(u, t)$ is a finite sum of integral monomials which, by Corollary 1 in §2, have order $> n$. If u is one of such monomials, then either $\gamma(u) > p$ or $\pi_p(u(u, t)) \equiv 0$. By Lemma 3, there exists a constant $C_1 > 0$ such that

$$|\pi_p(A(u, t))| \leq C_1 t^{n-p} \int_0^t |x_1(u, s)|^p ds \quad (6.12)$$

for all $u \in U$, $0 \leq t \leq 1$, $p = 1, \dots, \bar{p}$. Take $C_0 = 2C_1$. To determine a suitable t_0 , observe that $B(u, t)$ can be written as a finite sum of terms of the type

$$\begin{aligned} \Lambda(u, t) = \int_0^t \frac{1}{k!} G^{(k)}(0) (w_{p_1}(s), \dots, w_{p_l}(s), u_{l+1}(u, s), \dots, u_{l+h}(u, s), \\ x_1(u, s), \dots, x_1(u, s))u(s)ds, \end{aligned}$$

where $k \in \{1, \dots, n-1\}$, $l \geq 1$ and the integromials u_{l+1}, \dots, u_{l+h} have order > 1 . For $i = l+1, \dots, l+h$, let $p_i = \gamma(u_i)$. Observe that the order of u_i is then at least $p_i + 1$. If $z \in \bigcup_u (C_0, \tau)$ we thus have the inequalities

$$\begin{aligned} |u_1(u, s)| &\leq C \int_0^s |x_1(u, \sigma)|^{p_1} d\sigma, \\ |w_{p_1}(u, s)| &\leq C_0 s^{n-p_1} \int_0^s |x_1(u, \sigma)|^{p_1} d\sigma, \end{aligned}$$

for a suitable constant C and $s \leq \min\{\tau, 1\}$. The definition of S_p implies that either $p_1 + \dots + p_{l+h} + k - l - h \geq p$, or $\pi_p(\Lambda(u, t)) \equiv 0$. The same arguments used in the proof of Lemma 3 now yield constants $C_2, t_2 > 0$ such that

$$\pi_p(\Lambda(u, t)) \leq C_2 t^{n-p+1} \int_0^t |x_1(u, s)|^p ds, \quad (6.13)$$

for $0 < t < t_2$, $u \in U$.

Therefore, there exist $C_3, t_3 > 0$ such that

$$|\pi_p(B(u,t))| < C_3 t^{n-p+1} \int_0^t |x_1(u,s)|^p ds \quad (6.14)$$

for all $u \in U$, $p = 1, \dots, \bar{p}$, $t \in [0, t_3]$. Comparing (6.11) with (6.12) and (6.14), we see that Lemma 4 holds with $t_0 = \min \{1, t_3, C_1 C_3^{-1}\}$.

The conclusion of the proof of Theorem 2 is now straightforward. For all $u \in U$, $y_n(u, \cdot)$ is the unique fixed point of ψ_u . By Lemma 4, $y_n(u, \cdot) \in \mathcal{L}_u(c_0, t_0)$, hence (6.2) follows from (6.10).

With the same notation of Theorem 2 we have

Corollary 3. If $p > 1$, $n < p$, then there exist $C_0, t_0 > 0$ such that

$$|\pi_p(x(u,t) - x_n(u,t))| = |\pi_p(x(u,t))| < C_0 \int_0^t \left| \int_0^s G(\sigma) u(\sigma) d\sigma \right|^p ds \quad (6.15)$$

for all $u \in U$, $t \in [0, t_0]$, $p = 1, \dots, \bar{p}$.

Indeed, x_n is a sum of integral monomials u_1 having order $< n$. Hence $\gamma(u_1) < p$ and $\pi_p(u_1(u,t)) \equiv 0$. This implies $\pi_p(x_n(u,t)) \equiv 0$. Setting $n = p$, Theorem 2 yields the bound (6.15).

Example 4. consider on \mathbb{R}^3 the system $\dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) = (u \cos x_1 - x_2 - x_3, \operatorname{tg}(x_1 - x_3), \sin^2 x_1 - x_2^2 + x_1^3)$, $x(0) = 0 \in \mathbb{R}^3$. A third order expansion yields

$$x_1(u,t) = \int_0^t u(s) ds - \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} u(\sigma_3) d\sigma_3 d\sigma_2 d\sigma_1 + \epsilon_1(u,t),$$

$$x_2(u,t) = \int_0^t \int_0^{\sigma_1} u(\sigma_2) d\sigma_2 d\sigma_1 + \epsilon_2(u,t),$$

$$x_3(u,t) = \int_0^t \left(\int_0^s u(\sigma) d\sigma \right)^2 ds + \epsilon_3(u,t).$$

For this system, $S_1 = V_1 =$

$\{(\xi_1, \xi_2, 0) : \xi_1, \xi_2 \in \mathbb{R}\}$, $S_2 = \mathbb{R}^3$, $V_2 = \{(0, 0, \xi_3) : \xi_3 \in \mathbb{R}\}$. By Theorem 2 there exist $C, T > 0$ such that

$$|e_3(u,t)| \leq C t \int_0^t \left| \int_0^s u(\sigma) d\sigma \right|^2 ds$$

for all $u \in U$, $t \in [0, T]$. Hence, for small t , $x_3(u,t) > 0$ and the system is not locally controllable. An alternative proof of this could be obtained from the results in [6].

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ABSTRACT (Continued)

obtained by computing the n -th Picard iterate for the reduced system

$\dot{x}(t) = \sum_{i=1}^m (T^n g_i)(x(t)) u_i(t)$, $x(0) = 0$, discarding the terms of order $> n$. For $z \in \mathbb{R}^d$, directional error bounds of the form

$$| \langle z, x(u,t) - x_n(u,t) \rangle | \leq C t^{n-p} \int_0^t \left| \int_0^s \sum_{i=1}^m g_i(0) u_i(\sigma) d\sigma \right|^p ds$$

can be given. These estimates improve those supplied by the classical Taylor's theorem and yield results concerning local non-controllability.